Spatial-Temporal Union of Subspaces for Multi-body NRSFM: Supplementary Material

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Abstract. This paper provides a detailed derivation to the solutions and statistical data of the results provided in [2].

1 Detailed derivation of the solution

1.1 Solution for S

$$\mathtt{S} = \underset{\mathtt{S}}{\text{argmin}} \frac{1}{2}\|\mathtt{W} - \mathtt{RS}\|_{\mathtt{F}}^2 + < \mathtt{Y}_1, \mathtt{S}^\sharp - \mathtt{g}(\mathtt{S}) > + \frac{\beta}{2}\|\mathtt{S}^\sharp - \mathtt{g}(\mathtt{S})\|_{\mathtt{F}}^2 + < \mathtt{Y}_2, \mathtt{S} - \mathtt{SC}_1 >$$
$$+ \frac{\beta}{2}\|\mathtt{S} - \mathtt{SC}_1\|_{\mathtt{F}}^2.$$

We are minimizing this equation w.r.t $\mathtt{S}$. Therefore, we convert the second and third term in the above equation to the dimension of $\mathtt{S}$.

$\mathtt{S}^\sharp = \mathtt{g}(\mathtt{S}) \Rightarrow \mathtt{S} = \mathtt{g}^{-1}(\mathtt{S}^\sharp)$ (linear mapping).

Similarly, Lagrange multiplier $\mathtt{Y}_1$ is mapped to the dimension of $\mathtt{S}$.

$$\mathtt{S} = \underset{\mathtt{S}}{\text{argmin}} \frac{1}{2}\|\mathtt{W} - \mathtt{RS}\|_{\mathtt{F}}^2 + \frac{\beta}{2}\|\mathtt{g}^{-1}(\mathtt{S}^\sharp) - \mathtt{S}\|_{\mathtt{F}}^2 + < \mathtt{g}^{-1}(\mathtt{Y}_1), \mathtt{g}^{-1}(\mathtt{S}^\sharp) - \mathtt{S} >$$
$$< \mathtt{Y}_2, \mathtt{S} - \mathtt{SC}_1 > + \frac{\beta}{2}\|\mathtt{S} - \mathtt{SC}_1\|_{\mathtt{F}}^2.$$
$$= \underset{\mathtt{S}}{\text{argmin}} \frac{1}{2}\|\mathtt{W} - \mathtt{RS}\|_{\mathtt{F}}^2 + \frac{\beta}{2}(\|\mathtt{g}^{-1}(\mathtt{S}^\sharp)\|_{\mathtt{F}}^2 + \|\mathtt{S}\|_{\mathtt{F}}^2 - 2\mathtt{Tr}((\mathtt{g}^{-1}(\mathtt{S}^\sharp))^{\mathtt{T}}\mathtt{S}) +$$
$$\mathtt{Tr}((\mathtt{g}^{-1}(\mathtt{Y}_1))^{\mathtt{T}}(\mathtt{g}^{-1}(\mathtt{S}^\sharp))) - \mathtt{Tr}((\mathtt{g}^{-1}(\mathtt{Y}_1))^{\mathtt{T}}\mathtt{S}) + < \mathtt{Y}_2, \mathtt{S} - \mathtt{SC}_1 > + \frac{\beta}{2}\|\mathtt{S} - \mathtt{SC}_1\|_{\mathtt{F}}^2. \quad (1)$$
$$= \underset{\mathtt{S}}{\text{argmin}} \frac{1}{2}\|\mathtt{W} - \mathtt{RS}\|_{\mathtt{F}}^2 + \frac{\beta}{2}\left(\|\mathtt{S}\|_{\mathtt{F}}^2 - 2\mathtt{Tr}((\mathtt{g}^{-1}(\mathtt{S}^\sharp))^{\mathtt{T}}\mathtt{S}) - \frac{2}{\beta}\mathtt{Tr}((\mathtt{g}^{-1}(\mathtt{Y}_1))^{\mathtt{T}}\mathtt{S})\right) +$$
$$< \mathtt{Y}_2, \mathtt{S} - \mathtt{SC}_1 > + \frac{\beta}{2}\|\mathtt{S} - \mathtt{SC}_1\|_{\mathtt{F}}^2. \left\{\mathtt{S}^\sharp, \mathtt{Y}_1 \text{ are constants when minimizing over } \mathtt{S}\right\}$$

Since, adding constants to the above form will not affect the solution of S.

Therefore, we are adding $\|\mathtt{g}^{-1}(\mathtt{S}^\sharp) + \frac{\mathtt{g}^{-1}(\mathtt{Y}_1)}{\beta})\|_{\mathtt{F}}^2$ inside the second term,

which will give us the form

$$\mathtt{S} = \underset{\mathtt{S}}{\text{argmin}} \frac{1}{2}\|\mathtt{W} - \mathtt{RS}\|_{\mathtt{F}}^2 + \frac{\beta}{2}\|\mathtt{S} - (\mathtt{g}^{-1}(\mathtt{S}^\sharp) + \frac{\mathtt{g}^{-1}(\mathtt{Y}_1)}{\beta})\|_{\mathtt{F}}^2 + < \mathtt{Y}_2, \mathtt{S} - \mathtt{SC}_1 > +$$
$$\frac{\beta}{2}\|\mathtt{S} - \mathtt{SC}_1\|_{\mathtt{F}}^2.$$

The closed form solution for $\mathtt{S}$ can be derived by taking derivative of (1) w.r.t to $\mathtt{S}$ and equating to zero.

$$\frac{1}{\beta}(\mathtt{R}^{\mathtt{T}}\mathtt{R}+\beta\mathtt{I})\mathtt{S}+\mathtt{S}(\mathtt{I}-\mathtt{C}_1)(\mathtt{I}-\mathtt{C}_1^{\mathtt{T}}) = \frac{1}{\beta}\mathtt{R}^{\mathtt{T}}\mathtt{W}+\left(\mathtt{g}^{-1}(\mathtt{S}^{\sharp})+\frac{\mathtt{g}^{-1}(\mathtt{Y}_1)}{\beta}-\frac{\mathtt{Y}_2}{\beta}(\mathtt{I}-\mathtt{C}_1^{\mathtt{T}})\right). \tag{2}$$

1.2 Solution for $\mathtt{S}^{\sharp}$

$$\mathtt{S}^{\sharp} = \underset{\mathtt{S}^{\sharp}}{\text{argmin}} < \mathtt{Y}_1, \mathtt{S}^{\sharp}-\mathtt{g}(\mathtt{S}) > +\frac{\beta}{2}\|\mathtt{S}^{\sharp}-\mathtt{g}(\mathtt{S})\|_{\mathtt{F}}^2 + < \mathtt{Y}_3, \mathtt{S}^{\sharp}-\mathtt{S}^{\sharp}\mathtt{C}_2 > + \frac{\beta}{2}\|\mathtt{S}^{\sharp}-\mathtt{S}^{\sharp}\mathtt{C}_2\|_{\mathtt{F}}^2 + < \mathtt{Y}_8, \mathtt{S}^{\sharp}-\mathtt{J} > +\frac{\beta}{2}\|\mathtt{S}^{\sharp}-\mathtt{J}\|_{\mathtt{F}}^2.$$

Here, also the first two term and last two terms is condensed to a simpler form for mathematical convenience without affecting the final solution.

$$\mathtt{S}^{\sharp} = \underset{\mathtt{S}^{\sharp}}{\text{argmin}} \;\; \mathtt{Tr}\big(\mathtt{Y}_1^{\mathtt{T}}\mathtt{S}^{\sharp}\big) - \mathtt{Tr}\big(\mathtt{Y}_1^{\mathtt{T}}\mathtt{g}(\mathtt{S})\big) + \frac{\beta}{2}\big(\|\mathtt{S}^{\sharp}\|_{\mathtt{F}}^2 + \|\mathtt{g}(\mathtt{S})\|_{\mathtt{F}}^2 - 2\mathtt{Tr}((\mathtt{S}^{\sharp})^{\mathtt{T}}\mathtt{g}(\mathtt{S}))\big)$$
$$+ < \mathtt{Y}_3, \mathtt{S}^{\sharp}-\mathtt{S}^{\sharp}\mathtt{C}_2 > +\frac{\beta}{2}\|\mathtt{S}^{\sharp}-\mathtt{S}^{\sharp}\mathtt{C}_2\|_{\mathtt{F}}^2 + \mathtt{Tr}\big(\mathtt{Y}_8^{\mathtt{T}}\mathtt{S}^{\sharp}\big) - \mathtt{Tr}\big(\mathtt{Y}_8^{\mathtt{T}}\mathtt{J}\big) + \frac{\beta}{2}\big(\|\mathtt{S}^{\sharp}|_{\mathtt{F}}^2 + \|\mathtt{J}\|_{\mathtt{F}}^2 + $$
$$- 2\mathtt{Tr}\big((\mathtt{S}^{\sharp})^{\mathtt{T}}\mathtt{J}\big).$$

Since, we are minimizing over $\mathtt{S}^{\sharp}$. The terms which are not dependent on $\mathtt{S}^{\sharp}$ can be considered as constants, which gives us:

$$\mathtt{S}^{\sharp} = \underset{\mathtt{S}^{\sharp}}{\text{argmin}}\frac{\beta}{2}\big(\|\mathtt{S}^{\sharp}\|_{\mathtt{F}}^2 - 2\mathtt{Tr}(\mathtt{S}^{\sharp})^{\mathtt{T}}(\mathtt{g}(\mathtt{S})-\frac{\mathtt{Y}_1}{\beta})\big) + < \mathtt{Y}_3, \mathtt{S}^{\sharp}-\mathtt{S}^{\sharp}\mathtt{C}_2 > +\frac{\beta}{2}\|\mathtt{S}^{\sharp}-\mathtt{S}^{\sharp}\mathtt{C}_2\|_{\mathtt{F}}^2$$
$$+\frac{\beta}{2}\big(\|\mathtt{S}^{\sharp}\|_{\mathtt{F}}^2 - 2\mathtt{Tr}(\mathtt{S}^{\sharp})^{\mathtt{T}}(\mathtt{J}-\frac{\mathtt{Y}_8}{\beta})\big).$$

Adding $\|\mathtt{g}(\mathtt{S})-\frac{\mathtt{Y}_1}{\beta}\|_{\mathtt{F}}^2$ and $\|\mathtt{J}-\frac{\mathtt{Y}_8}{\beta}\|_{\mathtt{F}}^2$ inside the first term and last term respectively to get the quadratic form. As these terms are constants when minimizing over $\mathtt{S}^{\sharp}$ it will not affect the final solution.

$$\mathtt{S}^{\sharp} = \underset{\mathtt{S}^{\sharp}}{\text{argmin}}\frac{\beta}{2}\|\mathtt{S}^{\sharp}-(\mathtt{g}(\mathtt{S})-\frac{\mathtt{Y}_1}{\beta})\|_{\mathtt{F}}^2 + < \mathtt{Y}_3, \mathtt{S}^{\sharp}-\mathtt{S}^{\sharp}\mathtt{C}_2 > +\frac{\beta}{2}\|\mathtt{S}^{\sharp}-\mathtt{S}^{\sharp}\mathtt{C}_2\|_{\mathtt{F}}^2 + \frac{\beta}{2}\|\mathtt{S}^{\sharp}-(\mathtt{J}-\frac{\mathtt{Y}_8}{\beta})\|_{\mathtt{F}}^2. \tag{3}$$

The closed form solution for $\mathtt{S}^{\sharp}$ can be derived by taking derivative of (3) w.r.t $\mathtt{S}^{\sharp}$ and equating to zero.

$$\mathtt{S}^{\sharp}(2\mathtt{I}+(\mathtt{I}-\mathtt{C}_2)(\mathtt{I}-\mathtt{C}_2^{\mathtt{T}})) = \left(\mathtt{g}(\mathtt{S})-\frac{\mathtt{Y}_1}{\beta}\right)+(\mathtt{J}-\frac{\mathtt{Y}_8}{\beta})-\frac{\mathtt{Y}_3}{\beta}(\mathtt{I}-\mathtt{C}_2^{\mathtt{T}}). \tag{4}$$

1.3 Solution for $\mathtt{C}_1$

$$\mathtt{C}_1 = \underset{\mathtt{C}_1}{\operatorname{argmin}} < \mathtt{Y}_2, \mathtt{S} - \mathtt{S}\mathtt{C}_1 > + \frac{\beta}{2}\|\mathtt{S} - \mathtt{S}\mathtt{C}_1\|_\mathtt{F}^2 + < \mathtt{Y}_4, \mathtt{1}^\mathtt{T}\mathtt{C}_1 - \mathtt{1}^\mathtt{T} > +$$
$$\frac{\beta}{2}\|\mathtt{1}^\mathtt{T}\mathtt{C}_1 - \mathtt{1}^\mathtt{T}\|_\mathtt{F}^2 + < \mathtt{Y}_6, \mathtt{C}_1 - \mathtt{E}_1 > + \frac{\beta}{2}\|\mathtt{C}_1 - \mathtt{E}_1\|_\mathtt{F}^2. \tag{5}$$
$$= \underset{\mathtt{C}_1}{\operatorname{argmin}} \frac{\beta}{2}\|\mathtt{S}\mathtt{C}_1 - (\mathtt{S} + \frac{\mathtt{Y}_2}{\beta})\|_\mathtt{F}^2 + \frac{\beta}{2}\|\mathtt{1}^\mathtt{T}\mathtt{C}_1 - (\mathtt{1}^\mathtt{T} - \frac{\mathtt{Y}_4}{\beta})\|_\mathtt{F}^2 + \frac{\beta}{2}\|\mathtt{C}_1 - (\mathtt{E}_1 - \frac{\mathtt{Y}_6}{\beta})\|_\mathtt{F}^2.$$

The closed form solution for $\mathtt{C}_1$ is solved as:

$$(\mathtt{S}^\mathtt{T}\mathtt{S} + \mathtt{1}\mathtt{1}^\mathtt{T} + \mathtt{I})\mathtt{C}_1 = \mathtt{S}^\mathtt{T}(\mathtt{S} + \frac{\mathtt{Y}_2}{\beta}) + \mathtt{1}(\mathtt{1}^\mathtt{T} - \frac{\mathtt{Y}_4}{\beta}) + (\mathtt{E}_1 - \frac{\mathtt{Y}_6}{\beta}). \tag{6}$$

$$\mathtt{C}_1 = \mathtt{C}_1 - \operatorname{diag}(\mathtt{C}_1), \tag{7}$$

1.4 Solution for $\mathtt{C}_2$

$$\mathtt{C}_2 = \underset{\mathtt{C}_2}{\operatorname{argmin}} < \mathtt{Y}_3, \mathtt{S}^\sharp - \mathtt{S}^\sharp\mathtt{C}_2 > + \frac{\beta}{2}\|\mathtt{S}^\sharp - \mathtt{S}^\sharp\mathtt{C}_2\|_\mathtt{F}^2 + < \mathtt{Y}_5, \mathtt{1}^\mathtt{T}\mathtt{C}_2 - \mathtt{1}^\mathtt{T} > +$$
$$+ \frac{\beta}{2}\|\mathtt{1}^\mathtt{T}\mathtt{C}_2 - \mathtt{1}^\mathtt{T}\|_\mathtt{F}^2 + < \mathtt{Y}_7, \mathtt{C}_2 - \mathtt{E}_2 > + \frac{\beta}{2}\|\mathtt{C}_2 - \mathtt{E}_2\|_\mathtt{F}^2. \tag{8}$$
$$= \underset{\mathtt{C}_2}{\operatorname{argmin}} \frac{\beta}{2}\|\mathtt{S}^\sharp\mathtt{C}_2 - (\mathtt{S}^\sharp + \frac{\mathtt{Y}_3}{\beta})\|_\mathtt{F}^2 + \frac{\beta}{2}\|\mathtt{1}^\mathtt{T}\mathtt{C}_2 - (\mathtt{1}^\mathtt{T} - \frac{\mathtt{Y}_5}{\beta})\|_\mathtt{F}^2 + \frac{\beta}{2}\|\mathtt{C}_2 - (\mathtt{E}_2 - \frac{\mathtt{Y}_7}{\beta})\|_\mathtt{F}^2.$$

The closed form solution for $\mathtt{C}_2$ is derived as:

$$\left((\mathtt{S}^\sharp)^\mathtt{T}\mathtt{S}^\sharp + \mathtt{1}\mathtt{1}^\mathtt{T} + \mathtt{I}\right)\mathtt{C}_2 = (\mathtt{S}^\sharp)^T(S^\sharp + \frac{\mathtt{Y}_3}{\beta}) + \mathtt{1}(\mathtt{1}^\mathtt{T} - \frac{\mathtt{Y}_5}{\beta}) + (\mathtt{E}_2 - \frac{\mathtt{Y}_7}{\beta}). \tag{9}$$

$$\mathtt{C}_2 = \mathtt{C}_2 - \operatorname{diag}(\mathtt{C}_2), \tag{10}$$

1.5 Solution for $\mathtt{E}_1$

$$\mathtt{E}_1 = \underset{\mathtt{E}_1}{\operatorname{argmin}} \lambda_1\|\mathtt{E}_1\|_1 + \gamma_1\|\mathtt{E}_1\|_\mathtt{F}^2 + < \mathtt{Y}_6, \mathtt{C}_1 - \mathtt{E}_1 > + \frac{\beta}{2}\|\mathtt{C}_1 - \mathtt{E}_1\|_\mathtt{F}^2.$$
$$= \underset{\mathtt{E}_1}{\operatorname{argmin}} \lambda_1\|\mathtt{E}_1\|_1 + \gamma_1\|\mathtt{E}\|_\mathtt{F}^2 + \frac{\beta}{2}\|\mathtt{E}_1 - (\mathtt{C}_1 + \frac{\mathtt{Y}_6}{\beta})\|_\mathtt{F}^2.$$
$$= \underset{\mathtt{E}_1}{\operatorname{argmin}} \lambda_1\|\mathtt{E}_1\|_1 + \gamma_1\|\mathtt{E}_1\|_\mathtt{F}^2 + \frac{\beta}{2}\|\mathtt{E}_1\|_\mathtt{F}^2 - \beta < \mathtt{E}_1, (\mathtt{C}_1 + \frac{\mathtt{Y}_6}{\beta}) > \tag{11}$$
$$= \underset{\mathtt{E}_1}{\operatorname{argmin}} \lambda_1\|\mathtt{E}_1\|_1 + (\gamma_1 + \frac{\beta}{2})(\|\mathtt{E}_1\|_\mathtt{F}^2 + \frac{2\beta}{2\gamma_1 + \beta} < \mathtt{E}_1, \mathtt{C}_1 + \frac{\mathtt{Y}_6}{\beta} >).$$
$$= \underset{\mathtt{E}_1}{\operatorname{argmin}} \lambda_1\|\mathtt{E}_1\|_1 + (\gamma_1 + \frac{\beta}{2})\|\mathtt{E}_1 - \frac{\beta}{2\gamma_1 + \beta}(\mathtt{C}_1 + \frac{\mathtt{Y}_6}{\beta})\|_\mathtt{F}^2.$$

The closed form solution for $\mathtt{E}_1$ is reached as:

$$\mathtt{E}_1 = \mathcal{S}_{\frac{\lambda_1}{\gamma_1 + \frac{\beta}{2}}}(\frac{\beta}{2\gamma_1 + \beta}(\mathtt{C}_1 + \frac{\mathtt{Y}_6}{\beta})) \tag{12}$$

1.6 Solution for $\mathtt{E}_2$

The derivation for the solution of $\mathtt{E}_2$ is similar to the solution of $\mathtt{E}_1$.

$$\begin{aligned}\mathtt{E}_2 &= \underset{\mathtt{E}_2}{\operatorname{argmin}} \lambda_3 \|\mathtt{E}_2\|_1 + \gamma_3 \|\mathtt{E}_2\|_F^2 + < \mathtt{Y}_7, \mathtt{C}_2 - \mathtt{E}_2 > + \frac{\beta}{2} \|\mathtt{C}_2 - \mathtt{E}_2\|_{\mathtt{F}}^2 \\ &= \underset{\mathtt{E}_2}{\operatorname{argmin}} \lambda_3 \|\mathtt{E}_2\|_1 + (\gamma_3 + \frac{\beta}{2}) \|\mathtt{E}_2 - \frac{\beta}{2\gamma_3 + \beta} (\mathtt{C}_2 + \frac{\mathtt{Y}_7}{\beta})\|_{\mathtt{F}}^2.\end{aligned} \tag{13}$$

The closed form solution for $\mathtt{E}_2$ is reached as:

$$\mathtt{E}_2 = \mathcal{S}_{\frac{\lambda_3}{\gamma_3 + \frac{\beta}{2}}} \left(\frac{\beta}{2\gamma_3 + \beta} (\mathtt{C}_2 + \frac{\mathtt{Y}_7}{\beta}) \right). \tag{14}$$

2 Tables for each comparison

Table 1. Table corresponding to Figure 7

Datasets	BMM	PND	Zhu et al.	Kumar et al. [1]	Ours
Dance+Yoga	0.045	0.078	0.052	0.046	**0.043**
Drink+Walking	0.074	**0.060**	0.083	0.073	0.071
Shark+Stretch	0.024	**0.015**	0.067	0.025	0.019
Walking+Yoga	0.070	0.072	0.087	0.070	**0.066**
Face+Pickup	0.032	**0.012**	0.018	0.025	0.022
Face+Yoga	0.017	**0.010**	0.028	0.019	0.017
Shark+Yoga	0.035	**0.018**	0.094	0.037	0.033
Stretch+Yoga	0.039	0.109	0.045	0.039	**0.036**

Table 2. Table corresponding to Figure 8

Datasets	BMM	PND	Zhu et al.	Kumar et al. [1]	Ours
p2_free_2	0.1973	0.1544	**0.1142**	0.1992	0.1171
p2_grab_2	0.2018	0.1570	0.0960	0.2080	**0.0822**
p3_ball_1	0.1356	0.1477	0.0832	0.1348	**0.0810**
p4_meet_12	0.0802	0.0862	0.0972	0.0821	**0.0815**
p4_table_12	0.2313	0.1588	0.1322	0.2313	**0.0994**

Table 3. Table corresponding to Figure 11

Datasets	BMM	PND	Zhu et al.	Kumar et al. [1]	Ours
Face Sequence 1	0.078	0.077	0.082	0.075	**0.073**
Face Sequence 2	0.059	0.062	0.063	**0.050**	0.052
Face Sequence 3	0.042	0.051	0.057	**0.038**	0.039
Face Sequence 4	0.049	0.041	0.056	0.044	**0.040**

References

1. Kumar, S., Dai, Y., Li, H.: Multi-body non-rigid structure-from-motion. arXiv preprint arXiv:1607.04515 (2016)
2. Kumar, S., Dai, Y., Li, H.: Spatial-temporal union of subspaces for multi-body non-rigid structure-from-motion. arXiv preprint arXiv:1705.04916 (2017)